Remark on Kantorovich theorem and Urabe’s theorem

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ABSTRACT

Theoretical theorems are usually discussed on the real number system. But, in numerical computation, such system has to be approximated by real number system of a rather special type such as fixed point number system or floating point number system. Hence, when we discuss the theoretical theorems in numerical analysis, we must take care of their numerical properties.

**Introduction.** Let \( F(x) \) be a nonlinear function of an open subset \( D \) of \( R^* \) into \( R^* \). Let \( x^{(0)} \in D \) and \( U \) be a closed sphere of radius \( t^* \) centered at \( x^{(0)} \), i.e.,

\[
U = \{ x \in R^* ; ||x-x^{(0)}|| \leq t^* \}.
\]

The problem of finding a point \( x^* \in U \), where \( x^* \) is a unique solution of the equation \( F(x) = 0 \) can be attacked by Newton’s method that consists of generating a sequence \( \{x^{(m)}\} \) of successive approximations to \( x^* \) according to the algorithm

(1) \[
x^{(m+1)} = x^{(m)} - \frac{F(x^{(m)})}{F'(x^{(m)})}, \quad m = 0, 1, 2, ...
\]

starting from a preassigned guess \( x^{(0)} \in U \).

In our paper, we study two theorems : Kantorovich theorem and Urabe’s theorem that concern the error bound for the approximate solution obtained by the formula (1) and show that Urabe’s theorem gives numerically better approximation than that of Kantorovich theorem.

Before going to our example, we would like to state Kantorovich theorem and Urabe’s theorem.

**Kantorovich theorem (1948)**

Let \( F(x) \) be a differentiable function defined on the open domain \( D \subset R^* \) and let \( x^{(0)} \in D \).

Let us assume the following conditions :

(i) the Jacobian matrix \( J(x) \) of \( F(x) \) is Lipschitz continuous in \( D \):

\[
\| J(x) - J(y) \| \leq L \| x - y \| \quad (x, y \in D, L > 0),
\]

(ii) \( J(x^{(0)}) \) is regular, \( \| J^{-1}(x^{(0)}) \| \leq a, \| J^{-1}(x^{(0)}) F(x^{(0)}) \| \leq b \) and let \( h = abL \leq \frac{1}{2} \).

(iii) \( x^{(0)} \) is the center of the closed sphere \( U = \{ x \in R^* ; \| x-x^{(0)} \| \leq t^* \} \) of radius \( t^* = (1 - \sqrt{1-2h})/(aL) \) and \( U \subset D \).
At the same time we assume that the following conditions are also satisfied
(iv) \( x^* \) is a unique solution of \( F(x) = 0 \) in \( U \).
(v) \( x^{(0)} \) be the initial guess of the Newton sequence \( \{x^{(m)}\} \) and \( x^{(m)} \in U \).

Then

\[
(2) \quad \| x^{(m)} - x^* \| \leq \frac{(1 - \sqrt{1 - 2h})^m}{2^m aL} \quad (m = 0, 1, 2, \ldots).
\]

In case \( h = \frac{1}{2} \), the right-hand side of (2) becomes \( \frac{1}{2^m (aL)} \).

**Urabe’s theorem (1965)**

Let \( F(x) \) be a continuously differentiable function on the domain \( D \subset \mathbb{R}^n \). Let \( x^{(0)} \in D \) and suppose \( f(x^{(0)}) \) be regular. Also suppose that the following (i)-(iii) conditions are satisfied for a positive number \( \delta \) and a non-negative \( \kappa \) \((0 \leq \kappa < 1)\).

(i) \( \Omega_0 = \{ x \in \mathbb{R}^n : \| x - x^{(0)} \| \leq \delta \} \subset D \).

(ii) \( \| J(x) - J(x^{(0)}) \| \leq \frac{\kappa}{M} \quad (x \in \Omega_0) \).

(iii) \( \frac{M r}{1 - \kappa} \leq \delta \).

where \( \| F(x^{(0)}) \| \leq r \), \( \| f^{-1}(x^{(0)}) \| \leq M \).

Suppose \( x^* \in \Omega_0 \) is the unique solution of the equation \( F(x) = 0 \) and \( f(x^*) \) is regular. Then the error estimation

\[
(3) \quad \| x^{(0)} - x^* \| \leq \frac{Mr}{1 - \kappa}
\]

is satisfied.

**Our Example**

Let us consider the third degree polynomial equation \( F(x) = x^3 - 3x + 3 = 0 \). This equation has only one real root \( a = -2.103803402 \ldots \). Let \( x^{(0)} = -2.11 \) be the approximate solution.

**Application of Kantorovich theorem and Urabe’s theorem.**

Let \( z = x^{(0)} \). The Horner’s algorithm to calculate \( F(z) \) and \( F'(z) \) is given below:

\[
\begin{array}{cccc}
  k & 0 & 1 & 2 & 3 \\
  b^{(0)}[k] & 1 & 0 & -3 & 3 \\
  b^{(0)}[k-1]z & -2.11 & 4.45 & -3.06 \\
  b^{(0)}[k] & 1 & -2.11 & 1.45 & -0.06 \\
  b^{(0)}[k-1]z & -2.11 & 8.90 \\
  b^{(0)}[k] & 1 & -4.22 & 10.35 \\
\end{array}
\]

\[
F(z) = -0.06 \quad F'(z) = 10.35.
\]
[A](Kantorovich)
Suppose $D = [-3, -2].$

(i) $f(x) = 3x^2 - 3, \quad f(x) - f(y) = 3(x^2 - y^2) = 3(x - y)(x + y).$

As $\|f(x) - f(y)\| = 3 \|x + y\| \|x - y\|$, and $\|x + y\| \leq 6$ ($x, y \in D$),
\[ \|f(x) - f(y)\| \leq 18 \|x - y\|. \]

\[ \therefore \quad L = 18. \]

(ii) As $\|f^{-1}(z)\| = 1/|f'(z)| = 1/10.35 = 0.096618... \leq 0.0967$, we have
\[ a = 0.0967, \quad \text{and as } \|f^{-1}(z)f(z)\| = |f(z)/f'(z)| = 0.06/10.35 = 0.057971... \leq 0.0580. \]
\[ b = 0.00580. \quad \text{So, } h = abL = 0.0967 \times 0.00580 \times 18 = 0.010094... < \frac{1}{2}. \]

(iii) $\sqrt{1 - \frac{2h}{aL}} = \sqrt{1 - 0.0202} = 0.98985... = 0.990.$
\[ t^* = \frac{1 - \sqrt{1 - \frac{2h}{aL}}}{aL} = \frac{1 - 0.990}{0.0967 \times 18} = 0.0005745... . \]

As $t^* = 0.000575$, $U = \{x \in R : \|x - z\| \leq t^*\} = [-2.110575, -2.109425] \subset D = [-3, -2].$
So, all the conditions of Kantorovich theorem are satisfied. Thus the solution $x^* = a = -2.103803402...$ is the only one solution in the interval $U$, the error estimate is
\[ (K1) \quad \|x^* - z\| \leq t^* = 0.000575. \]
But this conclusion is not correct.

(E) $a \notin U$, and also $x^* - z = (-2.103803402...) - (-2.11) = 0.006196598... .$

Also, by Newton’s algorithm
\[ x^{(i)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})} = -2.11 - \frac{-0.06}{10.35} = -2.11 + 0.00597971 = -2.104202899. \]

So,
\[ x^{(i)} - x^* = -0.000399479. \]

On the other hand
\[ (K2) \quad \frac{1 - \sqrt{1 - \frac{2h}{aL}}}{2aL} = \frac{(1 - 0.990)^2}{2 \times 0.0967 \times 18} = 0.000028725. \]

Thus, by Kantorovich theorem the error estimation for $m = 1$ is also not correct.

[B](Urabe)
\[ r = \|f(z)\| = 0.06, \quad \|f^{-1}(z)\| = 1/|f'(z)| = 1/10.35 = 0.096618... \leq 0.0967 = M. \]

(i) In the closed domain $\Omega_i = \{x \in R : \|x - x^{(0)}\| \leq \delta\}$,
\[ x + z = (x - z) + 2z \quad \text{leads } \|x + z\| \leq 2 \|z\| + \delta \]
and
\[ \|f(x) - f(z)\| = 3 \|x + z\| \|x - z\| \leq (6 \|z\| + 3\delta)\delta. \]

So,

(ii) if $(6 \|z\| + 3\delta)\delta \leq \frac{\kappa}{M} (x \in \Omega_i)$ then
\[ \|f(x) - f(x^{(0)})\| \leq \frac{\kappa}{M} (x \in \Omega_i) \] is satisfied.

So, $\kappa \geq (6 \|z\| + 3\delta)\delta M = (12.66 + 3\delta)\delta \times 0.0967 = 1.224222\delta + 0.2901\delta^2.$

(iii) But $\frac{M \kappa}{1 - \kappa} \leq \delta$ is satisfied if
(4) \( \delta > \delta(1-\kappa) \geq M_r = 0.0967 \times 0.06 = 0.005802 \).
As the closed domain \( \Omega_\delta = \{ x \in R; \| x - x_r \| \leq \delta \} \) is the \( \delta \)-neighborhood of \( x = x_r = -2.11 \),
the value of \( \delta \) is negligible.
If we choose \( \delta = 0.10, \) then
\[ \Omega_\delta = \{ x \in R; \| x - (-2.11) \| \leq 0.10 \} = [-2.21, -2.01] \subset D = [-3, -2] \]
\[ \kappa \geq 1.24222(0.10) + 0.2901(0.10)^2 = 0.1253232 . \]
So, if we choose \( \kappa = 0.13, \) then all the conditions of Urabe's theorem are satisfied.
In the closed domain \( \Omega_\delta, \ x^* = \alpha = -2.10380... \) is the unique solution of \( F(x) = 0. \)
The error estimation
\[ \| z - x^* \| \leq \frac{M_r}{1 - \kappa} = \frac{0.0967 \times 0.06}{1 - 0.13} = 0.0066... \]
is satisfied.
The conclusion is correct because
\( x^* \in \Omega_\delta \) and \( z - x^* = -0.006196598... . \)

References

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